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Modeling traffic through a sequence of traffic lights

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We introduce a microscopic traffic model, based on kinematic behavior, which consists of a single vehicle traveling through a sequence of traffic lights that turn on and off with a specific frequency. The reconstructed function that maps the state of the vehicle from light to light displays complex behavior for certain conditions. This chaotic behavior, which arises by the discontinuous nature of the map, displays an essential ingredient in traffic patterns and could be of relevance in studying traffic situations.

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I. INTRODUCTION

The complex behavior displayed in traffic patterns is an interesting field of physics that is attracting some attention lately, in particular for their statistical [1,2] and dynamical [3,4] properties. There are a number references on traffic jams, chaotic traffic flows, bus-route problems, pedestrian flows, etc. [5–11].

In particular, the development of complex behavior in traffic flows determines, in a certain way, the efficiency of the transportation infrastructure of a city, region, or country. In this context, traffic flows, with and without passing, have been studied extensively in the literature [12,13]—e.g, cellular automaton models, mean-field theories which test the microscopic evolution, hydrodynamic models which approach collective behavior, etc. [14,15].

In this work, we are interested in the behavior of cars moving through a sequence of street light signals. Those who have been trapped in city traffic jams with traffic lights should understand the relevance of studying the dynamics of traffic patterns under these conditions and that the possibility of controlling these patterns may offer a solution to this very common problem. For this paper, we will concentrate on the behavior of a single car moving through a sequence of traffic lights, and we will see that under certain conditions unpredictable behavior arises. The understanding of this problem may help us approach the complex problem of interacting cars moving through a city with traffic lights.

II. MICROSCOPIC MODEL

The aim of our approach, although simplified, is to follow the details of one vehicle moving through a sequence of traffic lights in one dimension. The separation between the nth and $(n+1)$th traffic light is L_n. The nth light is green if $\sin(\omega_n t+\phi_n)>0$ and red otherwise, where ω_n is the frequency of the traffic light and ϕ_n is the time shift. Note that these two parameters are important if we were trying to control the traffic flow.

A car in this sequence of traffic lights can have (a) an acceleration a_+ until its velocity reaches the cruising speed $v_{\max}$, (b) a constant speed $v_{\max}$ with zero acceleration, or (c) a negative acceleration $-a_-$ until it stops; hence,

$$\frac{dv}{dt}=\begin{cases} a_+\theta(v_{\max}-v), & \text{accelerate},\\ -a_-\theta(v), & \text{brake},\end{cases}$$

where θ is the Heaviside step function.

As the car approaches the nth traffic light with velocity v the driver must make a decision—to step on the brakes or not—at the distance (the last stopping point) $v^2/2a_-$ depending on the sign of $\sin(\omega_n t+\phi_n)$. Note that if $(v_{\max}^2/2a_+)+(v_{\max}^2/2a_-)<L_n$, then $v=v_{\max}$ and the car reaches cruising speed before reaching the decision point. Also in general it makes sense that $(2\pi/\omega_n)>(v_{\max}/a_-),(v_{\max}/a_+)$ so that the traffic light does not change too fast from on to off. Of course as the vehicle brakes two things can happen: the car can stop completely and wait until the light turns on again or it can start accelerating before it stops completely if the light changes. Here we start observing the discontinuous nature of the model.

The car enters the sequence of traffic lights with velocity v_0 and time t_0. The set of rules described above determine a two-dimensional (2D) map $M(t_n,v_n)$ that evolves the state (t_n,v_n) at the nth traffic light to state (t_{n+1},v_{n+1}) at the $(n+1)$th traffic light. This map is constructed explicitly in the Appendix.

The types of trajectories between two traffic lights are described in Fig. 1, which clearly shows the typical kinematics associated with this model.

It is interesting to mention that this simplified model may still be relevant in the case of many cars going through the traffic light sequence, but with the effective parameters depending on the density of interacting cars. For example, you may have observed while driving through a city that the effective averaged acceleration seems to depend on the number of cars waiting at the traffic light. Similarly, the averaged effective cruising speed also seems to depend on the density of cars going through the sequence of traffic lights.

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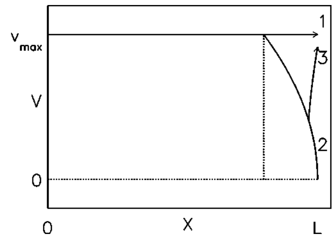

FIG. 1. The possible situations at the decision point—namely, (1) continuing, (2) braking to stop at $x=L$ before the light turns green again, and (3) braking and accelerating again as the light turns green before stopping completely.

III. ANALYSIS

We now study the situation of a car traveling through a sequence of N traffic lights, which in essence assumes a city with regular city blocks. We expect that iterating this map may reveal interesting information about the behavior of traffic flow in a city, even with this simplified model.

The best travel time occurs when the car speed synchronizes with the frequency of the traffic light; hence, $\sin(n\omega_n L_n/v_{\max}+\phi_n)>0$ for all n. This may be done for specific $v_{\max}$, but it cannot be guaranteed for a range of cruising speeds.

For now, we will concentrate on studying the dynamics for a given value of $v_{\max}$. Note that this parameter is very relevant in actual city situations since different drivers are willing to reach different values of $v_{\max}$ and traffic control strategies, achieved through ω_n and ϕ_n, will be very sensitive to its distribution. Furthermore, if we assume that the traffic parameters are, to first order, functions of the density or number of cars, then control strategies must take this into account especially during traffic jams.

Note that we could consider different $L_n=L+\Delta L_n$ values and different frequencies $\omega_n=\omega+\Delta\omega_n$ values as induced time phases $\Delta\phi_n=\omega\Delta L_n/v_{\max}$ and $\Delta\phi_n=\Delta\omega_n L/v_{\max}$, respectively. That is why we concentrate for simplicity on the situation $L_n=L$ and $\omega_n=\omega$. In this case it is convenient to define the cruising time as $T_c=L/v_{\max}$ and normalize $u=v/v_{\max}$, $\tau=t/T_c$, and $y=x/L$. The evolution equations then reduce to

$$\frac{du}{d\tau}=\begin{cases} A_+\theta(1-u), & \text{accelerate}, \\ -A_-\theta(u), & \text{brake}, \end{cases}$$

with $A_+=a_+L/v_{\max}^2$, $A_-=a_-L/v_{\max}^2$, and $\Omega=\omega T_c$.

The decision to stop or continue is made before the traffic signal at a distance

$$\Delta y=\frac{1}{2A_-},$$

depending on the sign of $\sin(\Omega\tau+\phi_n)$. The frequency restrictions reduce to $(2\pi/\Omega)>(1/A_+),(1/A_-)$. We propose to study the traffic flow as a function of A_+, A_-, and Ω. We define the acceleration ratio $a=A_+/A_-$. Initially we will take the phase $\phi_n=0$.

As the car, with a reasonable acceleration ratio $a=1/3$ and $A_+=10$ (corresponding to $T_c=L/v_{\max}>T_+=v_{\max}/a_+$), iterates through the traffic light sequence, we can observe that complex behavior appears for certain ranges of Ω. The case of $\Omega=6$ is shown in Figs. 2(a) and 2(b) for the speed u_n at the nth traffic light and time traveled $\Delta\tau_n=(\tau_{n+1}-\tau_n)$ between the nth and $(n+1)$th traffic lights, respectively. Clearly we observe a period-2 solution in which the car is caught by every other light, affecting the effective traffic flow.

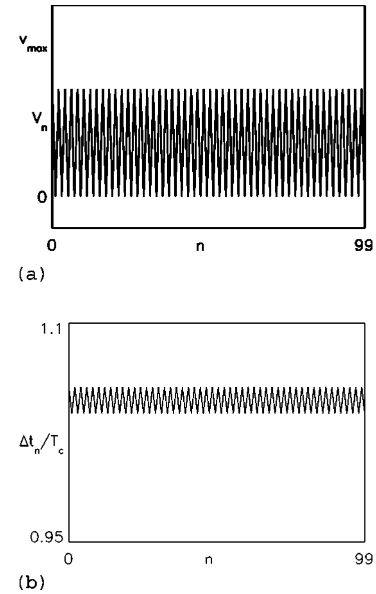

FIG. 2. The iterated map for (a) the speed u_n at the nth traffic light and (b) the normalized time traveled $\Delta t_n/T_c=(t_{n+1}-t_n)/T_c$ between traffic lights, for $\Omega=6$, $a=1/3$, and $A_+=10$.

Although all initial conditions in the u-τ plane reach this period-2 orbit asymptotically, a range of initial conditions reaches this orbit in one step due to the discontinuous nature of the map.

We then take a similar situation but with $\Omega=6.11$, and we observe the more complicated situation of Figs. 3(a) and 3(b) for u_n and $\Delta\tau_n$, respectively. Note that in this case, even though there exists a complex traffic behavior, the averaged traveling time is reduced, as compared with the situation of Fig. 2(b).

The bifurcation diagram in which we vary Ω is shown in Figs. 4(a) and 4(b) for the speed and time traveled between traffic lights. There is a particular range of frequencies where the iterated speed of the car varies in a complicated manner. Clearly, the average travel time in Fig. 2(b) has a larger value than the one for Fig. 3(b). In fact, it is worth noticing that the averaged travel time in the chaotic region coincides with the interpolation between the left and right nonchaotic regimes. But given the richer dynamics in the chaotic region, it could be possible to obtain a lower travel time through a chaos controlling strategy (e.g., Fouladi and Valdivia [16]). This will be explored elsewhere.

The bifurcation diagram of Fig. 4 suggests a period-doubling bifurcation to chaos as we increase Ω. As the chaotic attractor collides with one of the velocity thresholds, it produces an inverse period-doubling bifurcation. If we zoom into one of the frequency ranges where the map displays complex behavior, as shown in Fig. 5(a), we find an intricate structure of steady and chaotic behavior, as expected of a

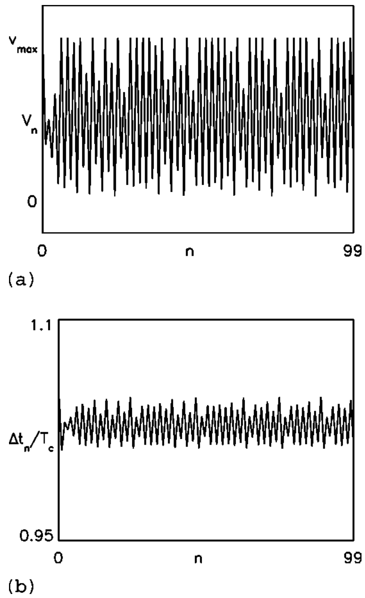

FIG. 3. The iterated map for (a) the speed u_n and (b) the time traveled $\Delta\tau_n$ between traffic lights, for $\Omega=6.11$, $a=1/3$, and $A_+=10$.

chaotic regime after a period-doubling bifurcation.

Estimating the relevance of this chaotic behavior and its sensitivity to perturbation and noise may be of importance in control strategies. In this sense a finite-amplitude Lyapunov exponent can be estimated [17]. Let us take a trajectory in the attractor that starts from (u_0, τ_0) and an initially perturbed trajectory that starts from $(u_0, \tau_0+\delta_0)$, with, for example, $\delta_0=10^{-7}$. The error is iterated n times, producing δ_n. Care must be taken to include only the scaling region where

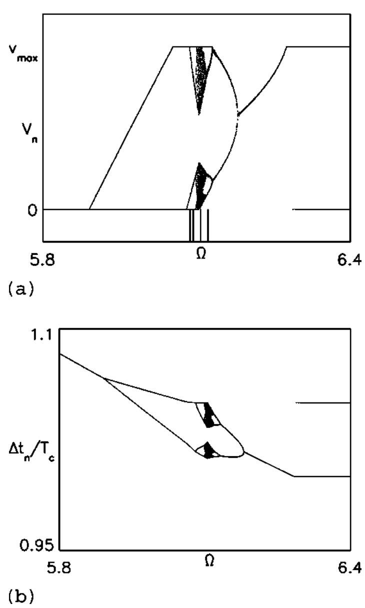

FIG. 4. The bifurcation diagram for (a) the speed and (b) the time between traffic light as a function of Ω. The other parameters are as before, $a=1/3$ and $A_+=10$. The transient has been removed. The four vertical lines at the bottom of (a) mark Ω values used to build Fig. 6.

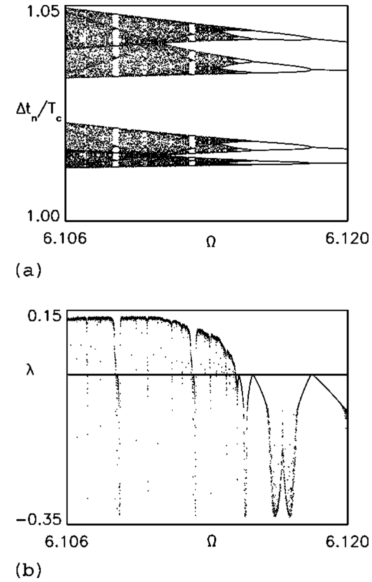

FIG. 5. The bifurcation diagram: (a) zoom for Fig. 4(b) and (b) the associated Lyapunov exponent.

$$\delta_n \sim \delta_0 e^{\lambda n}.$$

Given an initial condition over the attractor an exponent can be estimated by a fitting procedure in the scaling region. Of course, the discontinuous nature of the map complicates this calculation, where, for example, both trajectories can reach the same state in one step, yielding $\lambda=-\infty$. Nevertheless, a final Lyapunov exponent can still be constructed by averaging many initial conditions over the attractor, as shown in Fig. 5(b).

Another way to understand the dynamics of the system is to plot the phase space evolution, at a given value of Ω, as shown in Fig. 6 for four values of Ω. It is interesting to note that the dimension of the attractor is close to 1D in the chaotic situation. Here the volume contraction comes from the dynamics itself and the fact that a range of initial conditions goes to the same point in one iteration, hence its discontinuous nature.

The bifurcation diagram can be continued to larger values of Ω and windows with complex behavior similar to the one displayed in Fig. 4 can be found. In Fig. 7(a) we see the next window in a higher-frequency range. This case corresponds to increasingly faster light switching and may not be as relevant in actual traffic situations as the one described in Fig. 4.

Another parameter is $a=a_+/a_-$. In the limit $a\to 0$, with $a_-\to\infty$ and a_+ constant and finite, the driver makes the decision exactly at the traffic light and stops instantaneously if it is red. We expect that in this case the nature of the dynamics changes and any separation of trajectories in phase space—i.e., u-τ—can be understood in terms of the situation in which the cruising speed is synchronized with the traffic light. Figure 7(b) shows the bifurcation diagram for the car speed when $a=1/30$. The curves that resemble vertical lines correspond to the car not being synchronized with the traffic lights. These vertical lines should disappear as a is decreased

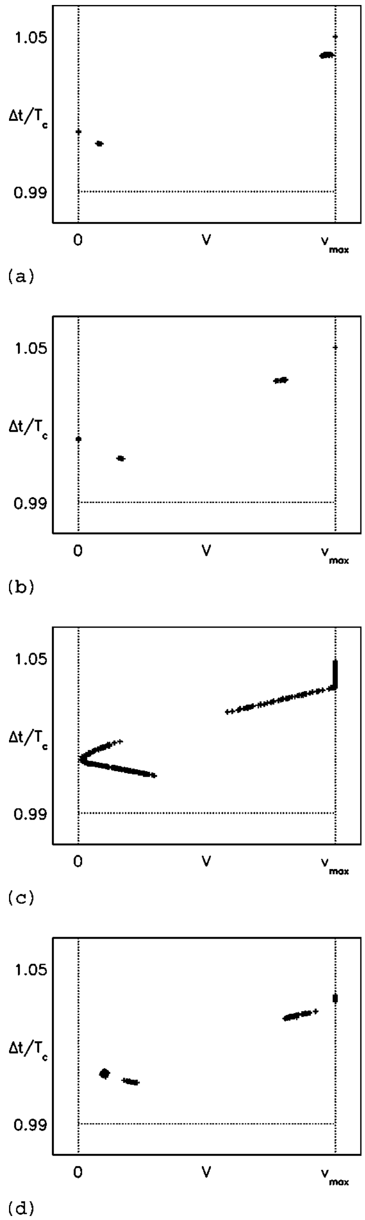

FIG. 6. The evolution in phase space, u-τ plane, at the four frequencies marked in Fig. 4(a) as vertical lines below $u=0$. The transient has been removed.

so that we obtain only two situations: either the car goes through the traffic light with $v_{\max}$ or stops completely.

We now turn to the problem of sensitivity with noise. We impose on the above model a random phase ϕ_n taken from a uniform distribution in $[0,0.01]$ and study the equivalent to Fig. 4. In Fig. 8, we can still observe the general bifurcation structure of Fig. 4 for v_n and $\Delta\tau_n$, but this structure is, of course, lost as we increase the perturbation amplitude.

IV. CONCLUSIONS

Suppose we design a traffic system—namely, ω_n and ϕ_n—so that it permits a continuous traffic flow for a certain cruising speed v_{max}. We can simulate the situation in which

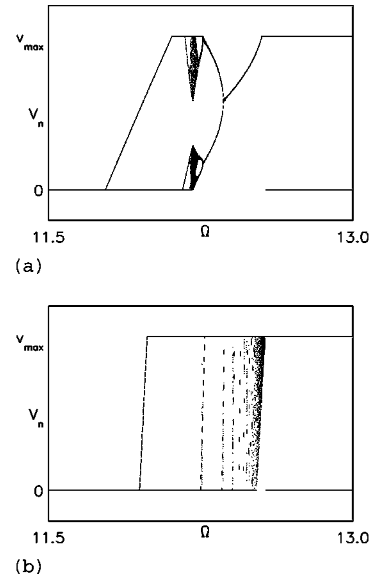

FIG. 7. (a) The bifurcation diagram for the speed similar to Fig. 4(a), but in a different frequency range. (b) The bifurcation diagram for the speed but for $a=1/30$.

the cruising speed is not exactly at the value the traffic lights were designed. This should be similar to the situation in which we vary Ω and study the bifurcation diagram as shown in Figs. 4(a) and 4(b) for the speed and time duration between traffic lights.

An important point to clarify is the relative size of the chaotic region which depends on the specific values of A_+, A_-, and L ($v_{\max}$ can be rescaled). Much larger chaotic regions than the ones illustrated here can be obtained by adjusting the parameters accordingly. For example, if we use $A_+=5$, $A_-=30$, and $T_c=10$, the chaotic regions are significantly larger from the ones previously shown and may apply in a

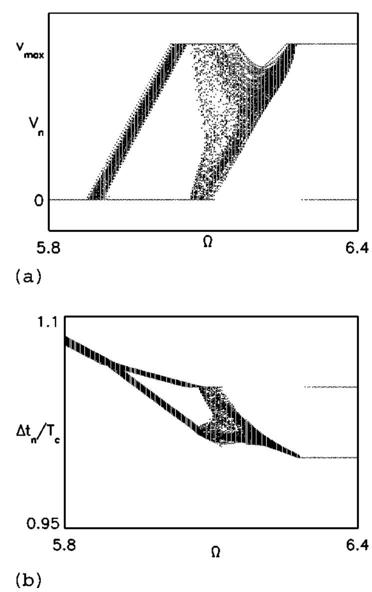

FIG. 8. (a) The bifurcation diagram including a random phase ϕ_n, for the case shown in Fig. 4. The noise is taken from a uniform random distribution between 0 and 0.01.

different traffic situation. Although the parameters used in this work were chosen to illustrate this ingredient in traffic dynamics without a specific city in mind, other parameter sets can represent a wide range of traffic conditions (street size, traffic light separation, car types, driver types, etc.). Note that ξ_d (see the Appendix) is another important variable determing the complexity in the behavior which defines the chaotic region.

Furthermore, if we want to extrapolate this model to the situation of more than one car, then A_+, A_-, and L may be obtained in a statistical sense from the distribution of parameters defining the traffic flow, distance between traffic lights, the car types, specific road situations, driver attitudes, etc. And in this case the values may be expected to be quite heterogeneous. Hence one could observe, for example, that a car moving in a traffic jam, accelerates and brakes all the time, contributing to an effective A_+ and A_-. This is an idea that may be of relevance for designing traffic flow during traffic jams.

It is worth noticing that the present analysis discusses the effect of long trips through the city, while short trips in a city would be affected by the transients in this model. Hence, this analysis points to the difficulties that may arise when trying to control the traffic flow in cities. With one car, we already have a complicated situation, and as we include more than one car we can only expect more interesting and complicated situations. Controlling such systems usually requires a control strategy that involves a large number of interacting agents.

Realistic situations are not as simple as the model we presented here; i.e., we have randomly varying street length L_n (or Ω_n or ϕ_n, etc.), a distribution of a_+ and a_-, etc. Some of these variations can be observed in a distribution of ϕ. In this sense it is worth mentioning that if we change ϕ_n randomly the chaotic behavior can be destroyed as expected for a large enough perturbation amplitude. On the other hand, if we chose the obvious deterministic phase $\phi_n = -\omega_n L_n / v_{\max}$ (other parameters kept constant), then the car can make it though the traffic sequence without stopping, hence optimal control. But this is not realistic because in practice the cars have a distribution of $v_{\max}$, and what works for one car, clearly does not work for another. If other deterministic functions are enforced, then interesting situations may also happen and will be studied elsewhere.

We clearly understand that the presented model is a strong simplification of actual traffic situations, but it keeps some essential features we believe are present in real traffic. This is just a very interesting starting point from which we can construct and interpret more complex scenarios—i.e., a work in progress.

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APPENDIX: THE $M(t,v)$ MAP

It is convenient to construct an exact map that relates successive crossing of the traffic lights. Let L be the distance between origin O and next traffic light. After crossing the nth light, the car reaches $v_{\max}$ at

$$x_c = \frac{v_{\max}^2 - v_n^2}{2a_+} \rightarrow y_c = \frac{1}{A_+}(1 - u_n^2),$$

$$t_c = t_n + \frac{v_{\max} - v_n}{a_+} \rightarrow \tau_c = \tau_n + \frac{1}{A_+}(1 - u_n),$$

$$v_c = v_{\max} \rightarrow u_c = 1,$$

and continues to move at constant velocity until the decision point

$$x_d = L_n - \frac{v_{\max}^2}{2a_-} \rightarrow y_d = 1 - \frac{1}{2A_-},$$

$$t_d = t_c + \frac{x_d - x_c}{v_{\max}} \rightarrow \tau_d = \tau_c + (y_d - y_c),$$

$$v_d = v_{\max} \rightarrow u_d = 1.$$

At this point we have two choices depending on the sign of $\sin(\omega_n t_d + \phi_n)$.

If $\sin(\omega_n t_d + \phi_n) = \sin(\Omega \tau_d + \phi_n) > 0$, the car reaches the traffic light with a state

$$x_{n+1} = L_n \rightarrow y_{n+1} = 1,$$

$$t_{n+1} = t_d + \frac{L_n - x_d}{v_{\max}} \rightarrow \tau_{n+1} = \tau_d + 1 - y_d,$$

$$v_{n+1} = v_{\max} \rightarrow u_{n+1} = 1.$$

If $\sin(\omega_n t_d + \phi_n) = \sin(\Omega \tau_d + \phi_n) < 0$, the car must start slowing down with a_-, and it will take an extra time $\Delta t = v_{\max}/a_-$ or $\Delta\tau = 1/A_-$ to reach the $(n+1)$th traffic light and stop. This time must be compared with the next time the light turns green, t_g, at which point the car can accelerate again. Defining the phase $\xi_d = \omega_n t_d + \phi_n = \Omega \tau_d + \phi_n$, we can compute

$$\xi_g = \omega_n t_g + \phi_n = 2\pi\left(\mathrm{Int}\left[\frac{\xi_d}{2\pi}\right] + 1\right),$$

where $\mathrm{Int}[x]$ is the integer part of x. Therefore, if $t_d + \Delta t < t_g$ or $\tau_d + \Delta\tau < \tau_g$, the car will cross the $(n+1)$th traffic light with

$$x_{n+1} = L_n \rightarrow y_{n+1} = 1,$$

$$t_{n+1} = t_g \rightarrow \tau_{n+1} = \tau_g,$$

$$v_{n+1} = 0 \rightarrow u_{n+1} = 0.$$

In the other case $t_d + \Delta t > t_g$ or $\tau_d + \Delta\tau > \tau_g$ the car starts accelerating at the state

$$x_g = x_d + v_d(t_g - t_d) - a_-(t_g - t_d)^2/2$$
$$\rightarrow y_g = y_d + u_d(\tau_g - \tau_d) - A_-(\tau_g - \tau_d)^2/2,$$

$$t_g = t_g \rightarrow \tau_g = \tau_g,$$

$$v_g = v_d - a_-(t_g - t_d) \rightarrow u_g = u_d - A_-(\tau_g - \tau_d),$$

and again we have two cases before it reaches L. We need to determine if the car reaches $v_{\max}$ before the light. We compute the distance at which the car reaches $v_{\max}$—namely, $x_m = x_g + (v_{\max}^2 - v_g^2)/2a_+$ or $y_m = y_g + (1-u_g^2)/2A_+$. Therefore, if $x_m > L$, then the car reaches the traffic light with

$$x_{n+1} = L_n \rightarrow y_{n+1} = 1,$$

$$t_{n+1} = t_g + \frac{v_{n+1} - v_g}{a_+} \rightarrow \tau_{n+1} = \tau_g + \frac{1}{A_+}(u_{n+1} - u_g),$$

$$v_{n+1} = \sqrt{v_g^2 + 2a_+(L_n - x_g)} \rightarrow u_{n+1} = \sqrt{u_g^2 + 2A_+(1 - y_g)};$$

otherwise, it reaches $v_{\max}$ at

$$x_m = x_m \rightarrow y_m = y_m,$$

$$t_m = t_g + \frac{v_{\max} - v_g}{a_+} \rightarrow \tau_m = \tau_g + \frac{1}{A_+}(1 - u_g),$$

$$v_m = v_{\max} \rightarrow u_f = 1,$$

and the light at

$$x_{n+1} = L_n \rightarrow y_{n+1} = 1,$$

$$t_{n+1} = t_m + \frac{L_n - x_m}{v_{\max}} \rightarrow \tau_{n+1} = \tau_m + (1 - y_m),$$

$$v_{n+1} = v_{\max} \rightarrow u_{n+1} = 1.$$

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